Tutorial 12

Let $\mathcal{A} = \{1, \cdots, n\}$ be the set of players and ν be a characteristic form.

Null player

Player i is said to be a null player of ν if

$$\nu(S \cup \{i\}) = \nu(S), \text{ for any } S \subseteq \mathcal{A} \setminus \{i\}.$$

Symmetric players

Two players i and j are said to be symmetric if

$$\nu(S \cup \{i\}) = \nu(S \cup \{j\}), \text{ for any } S \subseteq \mathcal{A} \setminus \{i, j\}.$$

Exercise 1 (The airport game)**.** *Building an airport will benefit n players. For each $i = 1, \cdots, n$, Player i requires an airport that costs c_i to build. To accommodate all the players, the airport should be built at a cost of* $\max_{1 \le i \le n} c_i$. *Suppose all the costs are distinct and $0 = c_0 < c_1 < \cdots < c_n$. Take the characteristic function to be*

$$\nu(S) = -\max_{i \in S} c_i.$$

For $k \in \{1, \cdots, n\}$, let $R_k = \{k, k+1, \cdots, n\}$ and define a function ν_k on $2^{\mathcal{A}}$ by

$$v_k(S) = \begin{cases} -(c_k - c_{k-1}) & \text{if } S \cap R_k \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Show that for each $k \in \{1, \cdots, n\}$, ν_k is a characteristic function.

(ii) Prove $\nu = \sum_{k=1}^{n} \nu_k$.

(iii) Show that for each $k \in \{1, \cdots, n\}$, if $i \notin R_k$, then Player i is a null player of ν_k.

(iv) Show that for each $k \in \{1, \cdots, n\}$, if $i, j \in R_k$, then Player i and Player j are symmetric players of ν_k.

(v) Find the Shapley values.

Solution. (i) is clear by checking the definition of characteristic function.

(ii) Let $S \subseteq \mathcal{A}$. If $S = \emptyset$, then clearly $\nu(S) = \sum_{k=1}^{n} \nu_k(S) = 0$. If $S \neq \emptyset$, let $k_0 = \max S$. Then by the definition of ν, $\nu(S) = -c_{k_0}$. On the other hand, note that $S \cap R_k \neq \emptyset$ for $k = 1, \cdots, k_0$ and $S \cap R_k = \emptyset$ for $k > k_0$. Hence $\nu_k(S) = -(c_k - c_{k-1})$ for $k = 1, \cdots, k_0$ and $\nu_k(S) = 0$ for $k > k_0$. Hence

$$\sum_{k=1}^{n} \nu_k(S) = \sum_{k=1}^{k_0} \nu_k(S) = \sum_{k=0}^{k_0} -(c_k - c_{k+1}) = -c_{k_0}.$$

We have proved that $\nu(S) = \sum_{k=1}^{n} \nu_k(S)$ for any $S \subseteq \mathcal{A}$, that is $\nu = \sum_{k=1}^{n} \nu_k$.

(iii) Let $S \subseteq \mathcal{A} \setminus \{i\}$ be arbitrary. Since $i \notin R_k$, we have

$$(S \cup \{i\}) \cap R_k = S \cap R_k,$$

which implies that $\nu_k(S \cup \{i\}) = \nu_k(S)$. That is Player i is a null player of v_k.

(iv) Let $S \subseteq \mathcal{A} \setminus \{i, j\}$ be arbitrary. Since $i, j \in R_k$, we have

$$\nu_k(S \cup \{i\}) = \nu_k(S \cup \{j\}) = -(c_k - c_{k-1}).$$

Hence Player i and Player j are symmetric players.

(v) For $i = 1$,

$$\phi_1 = \frac{1}{n!} \sum_{1 \in S \subseteq \mathcal{A}} (n - |S|)!(|S| - 1)!(\nu(S) - \nu(S \setminus \{1\})) = \frac{(n-1)!}{n!} \nu(\{1\}) = -\frac{c_1}{n}.$$

For $i = 2, \cdots, n$,

$$
\begin{aligned}
\phi_i &= \frac{1}{n!} \sum_{i \in S \subseteq \mathcal{A}} (n - |S|)!(|S|-1)!(\nu(S) - \nu(S \setminus \{i\})) \\
&= \frac{1}{n!} \sum_{i \in S \subseteq \mathcal{A}} (n - |S|)!(|S|-1)! \left[\sum_{k=1}^{n} (\nu_k(S) - \nu_k(S \setminus \{i\})) \right] \\
&= \frac{1}{n!} \sum_{i \in S \subseteq \mathcal{A}} (n - |S|)!(|S|-1)! \left[\sum_{k=1}^{i} (\nu_k(S) - \nu_k(S \setminus \{i\})) \right] \\
&= \frac{1}{n!} \sum_{S \subseteq \mathcal{A}:\ \max S = i} (n - |S|)!(|S|-1)! \left[\sum_{k=1}^{i} (\nu_k(S) - \nu_k(S \setminus \{i\})) \right] \\
&= -\frac{c_i}{n} + \frac{1}{n!} \sum_{S \subseteq \mathcal{A}:\ \max S = i, |S| \geq 2} (n - |S|)!(|S|-1)! \left[\sum_{k=1}^{i} (\nu_k(S) - \nu_k(S \setminus \{i\})) \right] \\
&= -\frac{c_i}{n} + \frac{1}{n!} \sum_{j=1}^{i-1} \sum_{S \subseteq \mathcal{A}:\ \max S = i, \max (S \setminus \{i\}) = j} (n - |S|)!(|S|-1)! \left[\sum_{k=j+1}^{i} -(c_k - c_{k-1}) \right] \\
&= -\frac{c_i}{n} - \frac{1}{n!} \sum_{j=1}^{i-1} (c_i - c_j) \sum_{S \subset \mathcal{A}:\ \max S = i, \max (S \setminus \{i\}) = j} (n - |S|)!(|S|-1)! \\
&= -\frac{c_i}{n} - \frac{1}{n!} \sum_{j=1}^{i-1} (c_i - c_j) \sum_{k=2}^{j+1} \sum_{S \subseteq \mathcal{A}:\ \max S = i, \max (S \setminus \{i\}) = j} (n-k)!(k-1)! \\
&= -\frac{c_i}{n} - \frac{1}{n!} \sum_{j=1}^{i-1} (c_i - c_j) \sum_{k=2}^{j+1} \binom{j-1}{k-2} (n-k)!(k-1)!.
\end{aligned}
$$

Exercise 2. *Let*

$$\mathcal{R} = \{(u, v) : (u-2)^2 + (v-2)^2 \leq 4\}.$$

Solve the Nash bargaining problem by using the following points as the status quo point (μ, ν).

(i) $(2, 2)$.

(ii) $(0, 2)$.

Solution. (i) The bargaining set is shown in Figure 1. Consider $g(u, v) =$

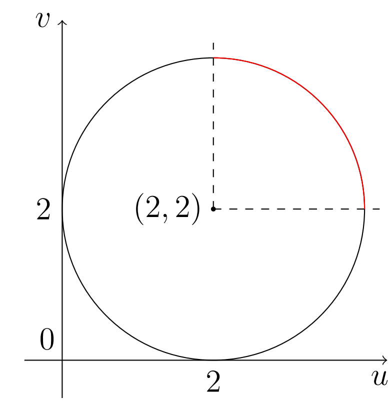

Figure 1

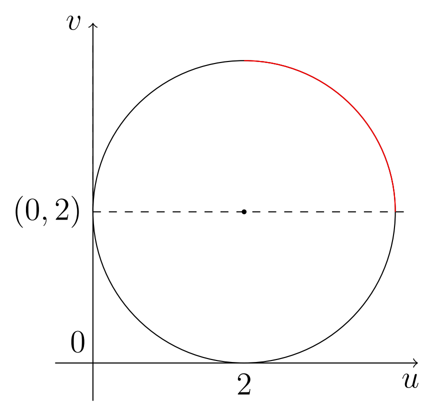

Figure 2

$(u-2)(v-2)$. On the bargaining set, $v = 2 + \sqrt{4-(u-2)^2}$. Hence

$$\begin{aligned} g(u,v) &= (u-2)(2+\sqrt{4-(u-2)^2}-2) \\ &= (u-2)(\sqrt{4-(u-2)^2}) \\ &\leq 2 \quad (\text{ by } 2ab \leq a^2+b^2). \end{aligned}$$

$g(u,v) = 2$ if and only if $u-2 = \sqrt{4-(u-2)^2}$, which implies that $u = 2+\sqrt{2}$. In this case, we have $v = 2+\sqrt{2}$. Hence the arbitration pair is $(2+\sqrt{2}, 2+\sqrt{2})$.

(ii) When the status point is $(0,2)$, the bargaining set is shown in Figure 2. In this case, on the bargaining set

$$g(u,v) = (u-0)(v-2) = u\sqrt{4-(u-2)^2}.$$

By elementary calculus, we see that g attains its maximum at $(u,v) = (3, 2+\sqrt{3})$. Hence the arbitration pair is $(3, 2+\sqrt{3})$.